PHYS 798C Spring 2022
Lecture 24 Summary

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I. SHAPIRO STEP DETAILS

Assume that the JJ is voltage biased (difficult to achieve in practice, but it simplifies the calculation considerably) as,
$V(t) = V_{dc} + V_{ac}\cos(\omega_{ac}t)$.
The gauge-invariant phase can be found by integrating the ac Josephson equation $\frac{d\gamma}{dt} = \frac{2e}{\hbar}V(t)$ as,
$\gamma(t) = \gamma(0) + \frac{2e}{\hbar}V_{dc}t + \frac{2e}{\hbar\omega_{ac}}V_{ac}\sin(\omega_{ac}t)$.
Define the Josephson frequency as $\omega_J \equiv \frac{2eV_{dc}}{\hbar}$.
In a typical experiment, one measures the time-averaged current $\langle I\rangle$ as a function of the dc bias voltage V_{dc}. Calculating the current in the RSJ-model junction (ignoring the capacitor) yields,
$I = \frac{V(T)}{R} + I_c\sin\left\{\gamma(0) + \omega_J t + \frac{2eV_{ac}}{\hbar\omega_{ac}}\sin(\omega_{ac}t)\right\}$.
Now use the important identity for the sine of the sine function (a part of the Jacobi-Anger relation):
$\sin(a + b\sin\theta) = \sum_{n=-\infty}^{\infty}(-1)^n J_n(b)\sin(a - n\theta)$, where $J_n(x)$ is the Bessel function, to find
$I(t) = \frac{V(T)}{R} + I_c\sum_{n=-\infty}^{\infty}(-1)^n J_n(\frac{2\pi V_{ac}}{\Phi_0\omega_{ac}})\sin\left[\gamma(0) + (\omega_J - n\omega_{ac})t\right]$.
Consider the time average of the current $\langle I\rangle$, which is the quantity usually measured in experiment. At an arbitrary driving frequency ω_{ac} the sine term will average to zero. However, at the special frequencies $\omega_J = n\omega_{ac}$ there will be a non-zero result,
$\langle I\rangle = \frac{V(T)}{R} + I_c\sum_{n=-\infty}^{\infty}(-1)^n J_n(\frac{2\pi V_{ac}}{\Phi_0\omega_{ac}})\sin\left[\gamma(0)\right]\delta_{\omega_J,n\omega_{ac}}$.
These dc voltage values produce a range of currents depending on the value of $\gamma(0)$, creating a series of current spikes periodic in V_{dc} riding on top of an Ohmic background. These are the Shapiro steps.
The above calculation predicts that the widths of the steps will be modulated with ac voltage amplitude V_{ac}. Due to the dependence of the Bessel functions for small arguments, $J_n(x) \sim x^n$ each step will appear in order as the microwave power is increased. The widths of the steps will change in a non-monotonic manner with increasing V_{ac}. The NIST voltage standard is based on a giant Shapiro step.

II. DC SQUIDS

A SQUID is a Superconducting QUantum Interference Device. The DC SQUID is a superconducting loop with a single Josephson junction on each branch of the loop. It acts as a sensitive magnetic flux to voltage transducer. The dc SQUID is current biased, and the voltage drop on the device is monitored as a function of magnetic flux in the SQUID loop.
The bias current splits two ways and can be written as,
$I_b = I_c\sin\gamma_1 + I_c\sin\gamma_2$, where it is assumed that both junctions have identical parameters (I_c, R, C), but their GIPD are different in general. The externally-imposed bias current forces γ_1 and γ_2 to change to satisfy this equation. Using a trigonometric identity, one can write the bias current as,
$I_b = 2I_c\cos\left(\frac{\gamma_1-\gamma_2}{2}\right)\sin\left(\frac{\gamma_1+\gamma_2}{2}\right)$.
Now we insist that the phase of the macroscopic quantum wavefunction be the same, modulo 2π upon completing a circuit C through the SQUID loop and coming back to the same point. Using the expressions for the GIPD at the two junctions, and the London relation between the current density, vector potential and gradient of the phase in the superconductors, one can derive the following result:
$\gamma_2 - \gamma_1 = 2\pi(n + \frac{\Phi}{\Phi_0})$, where $n = 0, \pm1, \pm2, ...$ and Φ is the magnetic flux in the entire SQUID loop. In detail, the flux $\Phi = \iint_S \vec{B}\cdot d\vec{S}$, S is any surface that terminates on the circuit C mentioned above.
One can use this to write the sum of the GIPDs as $\frac{\gamma_1+\gamma_2}{2} = \gamma_1 + \pi(n + \frac{\Phi}{\Phi_0})$.
With application of another trigonometric identity, we arrive at the result,

$$I_b = 2I_c\cos\left(\pi\frac{\Phi}{\Phi_0}\right)\sin\left(\gamma_1 + \pi\frac{\Phi}{\Phi_0}\right) = \widetilde{I}_c\sin\widetilde{\gamma}.$$

In other words, the dc SQUID acts as a single Josephson junction with a flux-tunable critical current. The renormalized critical current is $\widetilde{I}_c(\Phi) = 2I_c\cos\left(\pi\frac{\Phi}{\Phi_0}\right)$, the renormalized phase is given by

$\widetilde{\gamma} = \gamma_1 + \pi\frac{\Phi}{\Phi_0}$, the renormalized resistance is $\widetilde{R} = R/2$, and the renormalized capacitance is given by $\widetilde{C} = 2C$.
Note that in this derivation we assume that the "self-flux" produced by screening currents in the loop is small, or in other words $LI_c << \Phi_0$, where L is the self-inductance of the loop. We also assume that the superconducting films are thicker and wider than the penetration depth so that fluxoid quantization reduces to flux quantization for the circuit C.

The critical current of the SQUID is a periodic function of flux, repeating every time Φ advances through Φ_0. It ranges in value from $2I_c$ to zero, periodically as a function of flux in the SQUID loop. Consider the case of a SQUID with small capacitance. It will have an I-V curve given by,

$$\langle V \rangle = \begin{cases} 0 & I < \widetilde{I}_c \\ \widetilde{R}_N \sqrt{I^2 - \widetilde{I}_c^2(\Phi)} & I > \widetilde{I}_c \end{cases}$$

where $\widetilde{I}_c$ can be modulated between $2I_c$ and 0, depending on the flux applied to the SQUID. If we now bias the SQUID with a current just under $2I_c$, the voltage developed on the SQUID will be a function of flux applied. The dependence will be periodic in flux with period Φ_0, but not sinusoidal. The transfer function between voltage and flux is nonlinear, but can be linearized for small ranges of applied flux.

One can look at the DC SQUID as being analogous to a 2-slit interferometer for light. The critical current vs. flux has the appearance of a 2-slit diffraction pattern. Recall that putting magnetic flux through the barrier in a single junction creates the single-slit diffraction pattern for the junction critical current known as the Fraunhofer diffraction pattern. The DC SQUID creates two points (the Josephson junctions) that modulate the current through each branch of the loop by an amount that depends on γ_1 and γ_2. For half-integer flux values applied to the loop, the two junctions conspire to create zero net (super)-current through the device. Where does the current go if it is not getting through the DC SQUID? A similar question arises for currents going through an Aharonov-Bohm interferometer in mesoscopic transport. The answer is that the supercurrent can remain in the loop and circulate indefinitely there. As the flux through the loop is varied, this in turn varies the balance of super-current that flows through the device or circulates inside. In the context of nuclear physics, the circulating current is said to be entering a 'Feshbach mode,' while the current passing through takes advantage of a 'shape resonance' of the nucleus.